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ANALYSIS FOR PLASTIC BUCKLING OF THIN-WALLED CYLINDERS VIA NON-CLASSICAL CONSTITUTIVE THEORY OF PLASTICITY

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Abstract—The longitudinal compressive buckling loads of thin-walled cylinders in the yield region were analyzed by using both the incremental and the finite forms of a non-classical constitutive relation of plasticity (NCP). The relations of the critical stress σ_{cr} vs the ratio between R (the radius) and h (the thickness of the wall) were derived. The critical stresses of the thin-walled cylinders made of aluminum alloys AMΓ and Π1T were analyzed and compared with the experimental results. Comparison shows that for the cylinders made of Π1T both forms of NCP can provide reasonable prediction; but for those made of AMΓ, the result given by the finite form of NCP is satisfactory, while the critical stress predicted by the incremental form of NCP is about 25% higher than the experimental data. The capability of the NCP in the description of the material instability, the response of material under an abrupt strain disturbance and its easy application indicate that it may be of potential in the analysis of structural buckling.

INTRODUCTION

Thin-walled structures are widely used in aviation, chemistry, submarines, vehicles, nuclear reactors, civil engineering, and many other practical and high-technology industries. The failure of the structure caused by elastoplastic buckling has been extremely concerned over by the relevant researchers of mechanics and mechanical designers in a long period. In order to prevent this kind of failure, more realistic and reliable analysis is required. In the past several decades great effort has been made in this aspect, but few effective works can be found in the literature.

Since elastoplastic buckling may involve complicated loading/unloading, material instability and the effect of microscopic defects, it is necessary to use a more realistic constitutive relation for the analysis. In the past twenty years many new constitutive models, such as the multi-yield-surface model (Mroz, 1969), the tangent stiffness model (Dafalias, 1976), the model with the evolution of back stress that takes into account the thermal recovery effect (Chaboche, 1979), the endochronic theory of plasticity (Valanis, 1971, 1980) and non-classical constitutive theory of plasticity (NCP) (Fan, 1987; Fan and Peng, 1991; Peng and Fan, 1993), etc., have been proposed and have proved valid in many problems, but few results about their application in the analysis for elastoplastic buckling were reported.

In this paper, a non-classical relation of plasticity (NCP) is introduced based on a simple mechanical model and applied to the analysis of elastoplastic buckling of the thin-walled cylinder subjected to longitudinal compressive loading. It is seen that NCP can well describe the material instability and the response of the material under an abrupt strain disturbance, so one of the purposes of this work is to investigate whether or not the traditional "over-prediction" to the buckling stress by using the incremental form of the elastoplastic constitutive relation can be improved by using a sophisticated constitutive law. Both the incremental and finite forms of NCP were derived and applied to the elastoplastic buckling analysis. The critical stresses of the thin-walled cylinders made of aluminum alloys AMΓ and Π1T were analyzed and compared with the experimental data provided by Shen and Han (1981). Comparison shows that the results given by the finite form of NCP seems satisfactory while the incremental form of NCP may probably overpredict the critical stress in some cases.

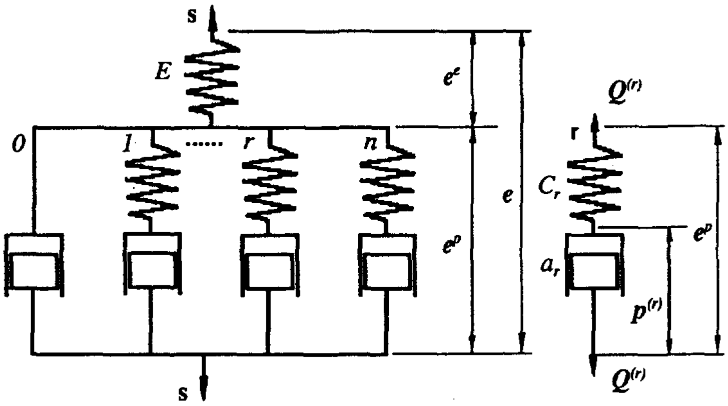

Fig. 1. A simple mechanical model for the proposed constitutive equation.

NON-CLASSICAL CONSTITUTIVE EQUATION OF PLASTICITY

In a series of papers, Fan (1987), Fan and Peng (1991), Peng and Ponter (1994) proposed a constitutive equation of plasticity for dissipative materials, which could be related to the Chaboche's model (Chaboche, 1979) and the endochronic constitutive equation (Valanis, 1980), but the microscopic aspects are further emphasized so that it may be able to describe the more complicated behavior of materials (Fan and Peng, 1991; Peng and Ponter, 1994).

The constitutive equation introduced in this paper is restricted to initially isotropic and plastically incompressible materials under the condition of isothermal and small deformation. For easy understanding, a simple mechanical model (see Fig. 1) is introduced. In Fig. 1, the r-th dissipative mechanism is described by the spring E_r (with stiffness C_r) and dashpot-like block a_r (with plastic damping factor a_r). E_r is related to stochastic internal structure on the microlevel and makes no contribution to the macroscopic elastic shear modulus G. The energy stored in the E_r corresponds to that stored in microstress field by the respective pattern of lattice defects, for instance, dislocation. From Fig. 1, it is obtained that

$$s_{ij} = Q_{ij}^{(0)} + \sum_{r=1}^{n} Q_{ij}^{(r)} \tag{1}$$

$$Q_{ij}^{(r)} = C_r(e_{ij}^p - p_{ij}^{(r)}) \tag{2}$$

with

$$e_{ij}^p = e_{ij} - \frac{s_{ij}}{2G} \tag{3}$$

where e_{ij}^p, e_{ij}^e and e_{ij} represent plastic, elastic and total deviatoric strains, respectively, s_{ij} denotes deviatoric stress and $p_{ij}^{(r)}$ and $Q_{ij}^{(r)}$ the r-th deviatoric internal variable and the corresponding generalized frictional force that satisfy the following relation

$$Q_{ij}^{(r)} = a_r \frac{\mathrm{d}p_{ij}^{(r)}}{\mathrm{d}\zeta} = a_r^0 f_r \frac{\mathrm{d}p_{ij}^{(r)}}{\mathrm{d}\zeta} \tag{4}$$

in which

$$\mathrm{d}\zeta = \| \mathrm{d}e_{ij}^p \| \tag{5}$$

where $\| \bullet \|$ denotes the Euclidean norm and f_r $(r = 0, 1, \ldots, n)$ the material hardening (softening) function that is closely related to the internal structure of deformed material. The evolution of $Q_{ij}^{(0)}$ can be expressed in the same form as that of $Q_{ij}^{(r)}$. If we choose $a_0^0 = s_y^0$ and notice that $p_{ij}^0 = e_{ij}^p$, then eqn (4) becomes

$$Q_{ij}^{(0)} = s_y^0 f_0 \frac{\mathrm{d}e_{ij}^p}{\mathrm{d}\zeta} \tag{6}$$

Combining eqn (2) and eqn (4), one obtains

$$\mathrm{d}Q_{ij}^{(r)} = C_r \mathrm{d}e_{ij}^p - \alpha_r f_r^{-1} Q_{ij}^{(r)} \mathrm{d}\zeta \quad (r = 1, 2, \ldots, n) \tag{7}$$

where $\alpha_r = C_r / a_r^0$. It is easily derived from eqns (1), (6) and (5) that

$$\left\| s_{ij} - \sum_{r=1}^{n} Q_{ij}^{(r)} \right\| - s_y^0 f_0 = 0 \tag{8}$$

If we choose $s_y^0 f_0 = k + R$ where k and R denote respectively initial yield stress and drag stress and $f_r = 1$ for $r = 1, 2, \ldots, n$, we reach Chaboche's model immediately. If $f_r = f(z)$ for $r = 0, 1, \ldots, n$ is chosen, the following endochronic constitutive relation is derived by substituting the integral of eqns (7) and (6) into eqn (1)

$$s_{ij} = s_y^0 \frac{\mathrm{d}e_{ij}^p}{\mathrm{d}z} + \int_0^z \rho(z - z') \frac{\mathrm{d}e_{ij}^p}{\mathrm{d}z'} \mathrm{d}z' \tag{9}$$

in which

$$\rho(z) = \sum_{r=1}^{n} C_r e^{-\alpha_r z} \tag{10}$$

and

$$\mathrm{d}z = \frac{\mathrm{d}\zeta}{f(z)} \tag{11}$$

as suggested by Wu and Yang (1983) and Wu *et al.* (1984). It, therefore, is able to encompass many other constitutive models as its special cases (Watanabe and Atluri, 1986; Chaboche, 1986).

It was found that for some materials there does not exist a distinct yield criterion. Inelasticity is a gradually developing process, the rate of which may be initially extremely small but increases with increasing loading. It was pointed out recently by Drucker that "the more sensitive the measurements that are made, the smaller will be the diameter of each yield surface. When a motion of a modest number of dislocations is detected as macroscopic plastic deformation the observed yield surface will shrink to zero" (Drucker, 1991). Assuming some kind of dissipative mechanism becomes active at the very beginning of deformation so that in eqn (9) s_y^0 (the size of the initial yield surface) tends to zero, the derived constitutive equation reduces to the following non-classical constitutive equation of plasticity (Peng and Fan, 1993)

$$s_{ij} = \int_0^z \rho(z - z') \frac{\mathrm{d}e_{ij}^p}{\mathrm{d}z'} \mathrm{d}z', \quad \text{or} \quad \mathrm{d}s_{ij} = \sum_{r=1}^{n} [C_r \mathrm{d}e_{ij}^p - \alpha_r Q_{ij}^{(r)} \mathrm{d}z] \tag{12}$$

On the other hand, it should be pointed out that eqn $(12)_1$ can also include eqn (9) as a special case provided $\alpha_1 \to \infty$ while $C_1/\alpha_1 < \infty$ so that the corresponding exponential function takes the unit-impulse function as its limit (Peng and Fan, 1993), which corresponds to $C_1 \to \infty$ (see Fig. 1). In practical application α_1 should be sufficiently large so that $\rho(z)$ becomes a highly decaying function which ensures the plastic strain rate to be very small in the vicinity of the origin. Both theoretical and numerical analyses showed that eqn (12) is able to well describe the material response under complex loading histories (Murakami and Read, 1987; Peng and Fan, 1992; Valanis and Fan, 1983; Fan and Peng, 1991).

In the case of monotonically proportional loading, eqn (12) leads to the following relation

$$s_{ij} = n_{ij}\int_0^z \rho(z-z')f(z')\,\mathrm{d}z' = A(z)n_{ij} \tag{13}$$

where n_{ij} stands for the unit vector along the loading direction and satisfies

$$n_{ij}n_{ij} = 1 \tag{14}$$

By defining

$$\mathrm{d}\bar{s} = \mathrm{d}s_{ij}n_{ij}, \quad \mathrm{d}\bar{e}^p = \mathrm{d}\zeta = \mathrm{d}e^p_{ij}n_{ij} \tag{15}$$

one has

$$\mathrm{d}e^p_{ij} = \left(\mathrm{d}\bar{e} - \frac{\mathrm{d}\bar{s}}{2G}\right)n_{ij}, \quad e^p_{ij} = \left(\bar{e}_{ij} - \frac{\bar{s}_{ij}}{2G}\right)n_{ij} \tag{16}$$

and the following relations

$$\mathrm{d}z = \frac{\mathrm{d}\zeta}{f(z)} = \frac{1}{f(z)}\left[\mathrm{d}\bar{e} - \frac{\mathrm{d}s}{2G}\right], \quad \zeta = B(z) = \bar{e} - \frac{\bar{s}}{2G} \tag{17}$$

where G denotes elastic shear modulus. Substituting eqn (17) into eqn (13) gives

$$\bar{s} = \frac{A(z)}{B(z)}\left(\bar{e} - \frac{\bar{s}}{2G}\right) \tag{18}$$

Letting σ and ε be the equivalent stress and strain, one can easily find that

$$\bar{e} = \sqrt{\tfrac{3}{2}}\bar{\varepsilon}, \quad \bar{s} = \sqrt{\tfrac{2}{3}}\bar{\sigma} \tag{19}$$

and therefore obtains the secant modulus by combining eqns (18) and (19)

$$E_s = \frac{\bar{\sigma}}{\bar{\varepsilon}} = \frac{3}{2}\frac{2GA(z)}{2GB(z)+A(z)} \tag{20}$$

Further, by differentiating eqn (13) and using eqn (17), we have

$$\mathrm{d}s_{ij} = n_{ij}A'(z)\,\mathrm{d}z = n_{ij}\frac{A'(z)}{f(z)}\left(\mathrm{d}\bar{e} - \frac{\mathrm{d}\bar{s}}{2G}\right) \tag{21}$$

so that the tangent modulus can be derived by combining eqns (15), (19) and (21) as

$$E_t = \frac{d\bar{\sigma}}{d\bar{\varepsilon}} = \frac{3}{2}\frac{2GA'(z)}{2Gf(z)+A'(z)} \tag{22}$$

Given the form of the hardening function $f(z)$, one can calculate the secant and tangent moduli respectively with eqns (20) and (22). For example, by choosing

$$f(z) = c-(c-1)\,e^{-\beta z} \tag{23}$$

as used by Wu and Yang (1983) and Wu *et al.* (1984), one obtains

$$A(z) = \sum_{r=1}^{n} \frac{C_r}{\alpha_r}\left[c(1-e^{-\alpha_r z}) - \frac{c-1}{\alpha_r-\beta}\alpha_r(e^{-\beta z}-e^{-\alpha_r z})\right]$$

$$B(z) = cz + \frac{c-1}{\beta}(e^{-\beta z}-1)$$

$$A'(z) = \sum_{r=1}^{n}\left[cC_r e^{-\alpha_r z} + \frac{c-1}{\alpha_r-\beta}C_r(\beta e^{-\beta z}-\alpha_r e^{-\alpha_r z})\right] \tag{24}$$

and then derives the corresponding E_s and E_t with eqns (20) and (22).

Now letting

$$E_s^0 = \frac{\sigma_x}{\varepsilon_x} \quad \text{and} \quad E_t^0 = \frac{d\sigma_x}{d\varepsilon_x} \tag{25}$$

be, respectively, the secant and tangent moduli at the point $(\sigma_x, \varepsilon_x)$ of monotonically tensile or compressive stress-strain curve, one can easily find the following relations

$$\frac{1}{E_s^0} = \frac{1-2v}{3E} + \frac{1}{E_s}, \quad \frac{1}{E_t^0} = \frac{1-2v}{3E} + \frac{1}{E_t} \tag{26}$$

in which v denotes Poisson's ratio.

THE BASIC EQUATIONS FOR THIN-WALLED CYLINDERS

In order to determine the critical stress corresponding to the initial elastoplastic buckling of thin-walled cylinders, we use the small deformation and plane stress assumptions, i.e., $\sigma_z = \tau_{xz} = \tau_{\theta z} = 0$. Suppose $\Delta\varepsilon_{x0}$, $\Delta\varepsilon_{\theta 0}$, $\Delta\gamma_{x\theta 0}$ and Δu, Δv, Δw be the increments of strain and displacement of the middle surface, then the strain increment of the point at z from the middle-surface can be expressed as (Wu and Xu, 1983)

$$\Delta\varepsilon_x = \Delta\varepsilon_{x0} - z\frac{\partial^2(\Delta w)}{\partial x^2}, \quad \Delta\varepsilon_\theta = \Delta\varepsilon_{\theta 0} - z\frac{\partial^2(\Delta w)}{R^2\partial\theta^2},$$

$$\Delta\gamma_{x\theta} = \Delta\gamma_{x\theta 0} - 2z\frac{\partial^2(\Delta w)}{R\,\partial x\,\partial\theta} \tag{27}$$

where

$$\Delta\varepsilon_{x0} = \frac{\partial(\Delta u)}{\partial x}, \quad \Delta\varepsilon_{\theta 0} = \frac{\partial(\Delta v)}{R\,\partial\theta} - \frac{\Delta w}{R}, \quad \Delta\gamma_{x\theta 0} = \frac{\partial(\Delta u)}{R\,\partial\theta} + \frac{\partial(\Delta v)}{\partial x} \tag{28}$$

The equilibrium equations of the material element of a thin-walled cylinder can be written as (Wu and Xu, 1983)

$$\frac{\partial(\Delta N_x)}{\partial x}+\frac{\partial(\Delta N_{x\theta})}{R\,\partial\theta}=0,\quad \frac{\partial(\Delta N_{x\theta})}{\partial x}+\frac{\partial(\Delta N_\theta)}{R\,\partial\theta}=0$$

$$\frac{\partial^2(\Delta M_x)}{\partial x^2}+2\frac{\partial^2(\Delta M_{x\theta})}{R\,\partial x\,\partial\theta}+\frac{\partial^2(\Delta M_\theta)}{R^2\partial\theta^2}+\frac{\Delta N_\theta}{R}+ h\left(\sigma_x\frac{\partial^2(\Delta w)}{\partial x^2}+\sigma_\theta\frac{\partial^2(\Delta w)}{R^2\partial\theta^2}+2\tau_{x\theta}\frac{\partial^2(\Delta w)}{R\,\partial\theta\partial x}\right)=0 \quad (29)$$

in which σ_x, σ_θ and $\tau_{x\theta}$ represent the stress at middle-surface before buckling occurs, ΔN_x, ΔN_θ, $\Delta N_{x\theta}$ and ΔM_x, ΔM_θ, $\Delta M_{x\theta}$ denote respectively the increments of the traction and moment on the cross section and are defined as

$$\Delta N_x=\int_{-h/2}^{h/2}\Delta\sigma_x\,\mathrm{d}z,\quad \Delta N_\theta=\int_{-h/2}^{h/2}\Delta\sigma_\theta\,\mathrm{d}z,\quad \Delta N_{x\theta}=\int_{-h/2}^{h/2}\Delta\tau_{x\theta}\,\mathrm{d}z$$

$$\Delta M_x=\int_{-h/2}^{h/2}\Delta\sigma_x z\,\mathrm{d}x,\quad \Delta M_\theta=\int_{-h/2}^{h/2}\Delta\sigma_\theta z\,\mathrm{d}z,\quad \Delta M_{x\theta}=\int_{-h/2}^{h/2}\Delta\tau_{x\theta}z\,\mathrm{d}z \quad (30)$$

where h is the thickness of the wall of the cylinder. If a stress function ϕ is introduced such that

$$\Delta N_x=\frac{\partial^2\phi}{R^2\partial\theta^2},\quad \Delta N_\theta=\frac{\partial^2\phi}{\partial x^2},\quad \Delta N_{x\theta}=-\frac{\partial^2\phi}{R\,\partial x\,\partial\theta} \quad (31)$$

then the first two equations of (29) will always be satisfied, and this problem can thus be solved by making use of the third equation of (29), the constitutive equation and the following compatibility equation obtained by eliminating Δu and Δv from eqn (28)

$$\frac{\partial^2(\Delta\varepsilon_{x0})}{R^2\partial\theta^2}+\frac{\partial^2(\Delta\varepsilon_{\theta0})}{\partial x^2}-\frac{\partial^2(\Delta\gamma_{x\theta0})}{R\,\partial\theta\partial x}+\frac{\partial^2(\Delta w)}{R\,\partial x^2}=0 \quad (32)$$

CRITICAL STRESS ANALYZED BY INCREMENTAL FORM OF NCP

Peng and Fan (1993) proposed the following incremental form of the non-classical constitutive equation

$$\Delta s_{ij}=2G_p\Delta e_{ij}+T_pB_{ij}\Delta z \quad (33)$$

in which

$$T_p=\left(1+\frac{A}{2G}\right)^{-1},\quad 2G_p=AT_p$$

$$A=\sum_{r=1}^{3}k_rC_r,\quad B_{ij}=-\sum_{r=1}^{3}k_r\alpha_r s_{ij}^{(r)}(z_n),\quad k_r=\frac{1-\mathrm{e}^{-\alpha_r\Delta z}}{\alpha_r\Delta z}$$

$$s_{ij}^{(r)}(z_n)=\mathrm{e}^{-\alpha_r\Delta z}s_{ij}^{(r)}(z_{n-1})+\frac{\Delta e_{ij}^pC_r}{\Delta z\alpha_r}(1-\mathrm{e}^{-\alpha_r\Delta z}) \quad (34)$$

In the case of plane stress, eqn (33) can be expressed as (Peng and Fan, 1993)

$$\begin{Bmatrix} \Delta\sigma_x \\ \Delta\sigma_\theta \\ \Delta\tau_{x\theta} \end{Bmatrix} = \begin{bmatrix} D'_{11} & D'_{12} & D'_{14} \\ D'_{21} & D'_{22} & D'_{24} \\ D'_{41} & D'_{42} & D'_{44} \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_x \\ \Delta\varepsilon_\theta \\ \Delta\gamma_{x\theta} \end{Bmatrix} \tag{35}$$

in which

$$D'_{ij} = \frac{D_{i3}D_{3j}}{D_{33}} \quad (i,j = 1,2,4) \tag{36}$$

$$[D] = [D_1] + \frac{2(G-G_p)}{H}[D_2], \quad H = 1 + \frac{T_p}{2Gf^2(z)\Delta z} B_{ij}\Delta e^p_{ij}$$

$$[D_1] = \begin{bmatrix} C_1 & C_2 & C_2 & 0 \\ C_2 & C_1 & C_2 & 0 \\ C_2 & C_2 & C_1 & 0 \\ 0 & 0 & 0 & G_p \end{bmatrix}, \quad C_1 = K + \tfrac{4}{3}G_p, \quad C_2 = K - \tfrac{2}{3}G_p$$

$$[D_2] = \frac{T_p}{2G_p f^2(z)\Delta z}(B_x, B_\theta, B_r, B_{x\theta})^T(\Delta e^p_x, \quad \Delta e^p_\theta, \quad \Delta e^p_r, \quad \Delta e^p_{x\theta}) \tag{37}$$

Integrating eqn (35) over the thickness h and noticing eqn (27) one obtains

$$\begin{Bmatrix} h\Delta\varepsilon_{x0} \\ h\Delta\varepsilon_{\theta 0} \\ h\Delta\gamma_{x\theta 0} \end{Bmatrix} = \begin{bmatrix} D'_{11} & D'_{12} & D'_{14} \\ D'_{21} & D'_{22} & D'_{24} \\ D'_{41} & D'_{42} & D'_{44} \end{bmatrix}^{-1} \begin{Bmatrix} \Delta N_x \\ \Delta N_\theta \\ \Delta N_{x\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{14} \\ C_{21} & C_{22} & C_{24} \\ C_{41} & C_{42} & C_{44} \end{bmatrix} \begin{Bmatrix} \Delta N_x \\ \Delta N_\theta \\ \Delta N_{x\theta} \end{Bmatrix} \tag{38}$$

$$\begin{Bmatrix} \Delta M_x \\ \Delta M_\theta \\ \Delta M_{x\theta} \end{Bmatrix} = -\frac{h^3}{12} \begin{bmatrix} D'_{11} & D'_{12} & D'_{14} \\ D'_{21} & D'_{22} & D'_{24} \\ D'_{41} & D'_{42} & D'_{44} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2(\Delta w)}{\partial x^2} \\ \dfrac{\partial^2(\Delta w)}{R^2\partial\theta^2} \\ 2\dfrac{\partial^2(\Delta w)}{R\,\partial x\,\partial\theta} \end{Bmatrix} \tag{39}$$

When a thin-walled cylinder is subjected to longitudinal compressive load and before buckling occurs, the stress state can be expressed as

$$\sigma_\theta = \tau_{x\theta} = 0, \quad \sigma_x = -\frac{P}{2\pi Rh} \tag{40}$$

In buckling analysis the assumption of geometrical imperfection is important. It was reported (Shen and Han, 1981) that when subjected to longitudinal compressive load, the initial elastoplastic buckling of the thin-walled cylinder is axisymmetrical and develops gradually, so it is reasonable to assume the following form of geometrical imperfection

$$\Delta w = w_m \sin\left(\frac{m\pi}{L}x\right) \tag{41}$$

Correspondingly, the stress function ϕ can be chosen as follows

$$\phi = \phi_m \sin\left(\frac{m\pi}{L}x\right) \tag{42}$$

Although in some of the previous work (Wu and Xu, 1983), both eqns (41) and (42) were used to formulate the critical stress, it can be found that eqn (42) is sufficient to take into account the effect of the geometrical imperfection, while eqn (41) can be derived as a result of eqn (42). By using the relation in eqn (31) the following generalized incremental stress field corresponding to the given geometrical imperfection can be obtained

$$\Delta N_x = \Delta N_{x\theta} = 0, \quad \Delta N_\theta = -\left(\frac{m\pi}{L}\right)^2 \phi_m \sin\left(\frac{m\pi}{L}x\right) \tag{43}$$

With eqns (38) and (28), one obtains the following incremental strain field in the middle-surface

$$\begin{Bmatrix} \Delta\varepsilon_{x0} \\ \Delta\varepsilon_{\theta 0} \\ \Delta\gamma_{x\theta 0} \end{Bmatrix} = -\frac{1}{h}\left(\frac{m\pi}{L}\right)^2 \phi_m \sin\left(\frac{m\pi}{L}x\right) \begin{Bmatrix} C_{12} \\ C_{22} \\ C_{42} \end{Bmatrix} \tag{44}$$

and the component of displacement, Δw, by noticing that the initial buckling is assumed to be axisymmetrical,

$$\Delta w = \frac{R}{h}\left(\frac{m\pi}{L}\right)^2 \phi_m C_{22} \sin\left(\frac{m\pi}{L}x\right) \tag{45}$$

This is just the assumed geometrical imperfection shown in eqn (41) provided

$$w_m = \frac{R}{h}\phi_m C_{22}\left(\frac{m\pi}{L}\right)^2 \tag{46}$$

By substituting eqn (45) into eqn (39), one obtains the following relation

$$\begin{Bmatrix} \Delta M_x \\ \Delta M_\theta \\ \Delta M_{x\theta} \end{Bmatrix} = \frac{h^2 R}{12}\left(\frac{m\pi}{L}\right)^4 \phi_m \sin\left(\frac{m\pi}{L}x\right) \begin{Bmatrix} D'_{12} \\ D'_{22} \\ D'_{42} \end{Bmatrix} \tag{47}$$

It can easily be found that the derived strain and displacement fields (see eqns (44) and (45)) satisfy the compatibility eqn (32).

Making use of the third relation in eqn (29) as well as eqns (43), (45) and (47), one obtains the following compressive stress corresponding to initial buckling

$$\sigma_x = -\left[\frac{h^2 D'_{11}}{12}\left(\frac{m\pi}{L}\right)^2 + \frac{1}{R^2 C_{22}}\left(\frac{L}{m\pi}\right)^2\right] \tag{48}$$

The critical condition for buckling to occur is that the work determined by the given generalized stress and the corresponding strain field is equal to the corresponding external work, i.e.,

$$\Delta w_{int} = \Delta w_{ext} \tag{49}$$

which leads to

$$h\int_0^L \sigma_x \Delta\varepsilon_{x0}\,\mathrm{d}_x + \int_0^L \Delta N_\theta \Delta\varepsilon_{\theta 0}\,\mathrm{d}x + \int_0^L \Delta M_x \frac{\partial^2(\Delta w)}{\partial x^2}\mathrm{d}x = -\frac{P\Delta u}{2\pi R} \tag{50}$$

By making use of eqns (28), (40), (43), (44) and (45), it is derived that

$$\left(\frac{m\pi}{L}\right)^4 = \frac{12}{R^2h^2}\frac{1}{C_{22}D'_{11}} \tag{51}$$

This is just another form of the critical condition expressed in eqn (49), from which the wave length L/m can be determined. The combination of this critical condition and eqn (48) gives the following critical stress σ_{cr} corresponding to initial buckling

$$\sigma_{cr} = |\sigma_x|_{min} = \frac{h}{\sqrt{3}R}\sqrt{\frac{D'_{11}}{C_{22}}} \tag{52}$$

For convenience of calculation, eqn (52) can be rewritten as

$$\frac{R}{h} = \frac{1}{\sqrt{3}\sigma_{cr}}\sqrt{\frac{D'_{11}}{C_{22}}} \tag{53}$$

For any given σ_{cr}, the matrixes **[D]**, **[D′]** and **[C]** can be calculated sequentially and then the corresponding R/h can easily be determined.

CRITICAL STRESS ANALYZED BY FINITE FORM OF NCP

In the deformation theory of plasticity, the relation between stress and strain of plane stress problem can be expressed as

$$\begin{aligned}
\varepsilon_x &= \frac{1-2v}{3E}(\sigma_x+\sigma_\theta) + \frac{1}{E_s}(\sigma_x - \tfrac{1}{2}\sigma_\theta) \\
\varepsilon_\theta &= \frac{1-2v}{3E}(\sigma_x+\sigma_\theta) + \frac{1}{E_s}(\sigma_\theta - \tfrac{1}{2}\sigma_x) \\
\gamma_{x\theta} &= \frac{3}{E_s}\tau_{x\theta}
\end{aligned} \tag{54}$$

and the corresponding increments of strain components can be derived as

$$\begin{aligned}
\Delta\varepsilon_x &= K_0(\Delta\sigma_x+\Delta\sigma_\theta) + \frac{1}{E_s}(\Delta\sigma_x - \tfrac{1}{2}\Delta\sigma_\theta) + K_1(\sigma_x - \tfrac{1}{2}\sigma_\theta)\frac{\Delta\bar{\sigma}}{\bar{\sigma}} \\
\Delta\varepsilon_\theta &= K_0(\Delta\sigma_x+\Delta\sigma_\theta) + \frac{1}{E_s}(\Delta\sigma_\theta - \tfrac{1}{2}\Delta\sigma_x) + K_1(\sigma_\theta - \tfrac{1}{2}\sigma_x)\frac{\Delta\bar{\sigma}}{\bar{\sigma}} \\
\Delta\gamma_{x\theta} &= \frac{3}{E_s}\Delta\tau_{x\theta} + K_1\frac{3\tau_{x\theta}}{\bar{\sigma}}\Delta\bar{\sigma}
\end{aligned} \tag{55}$$

in which

$$\begin{aligned}
\bar{\sigma} &= \sqrt{\sigma_x^2+\sigma_\theta^2-\sigma_x\sigma_\theta+3\tau_{x\theta}^2} \\
\Delta\bar{\sigma} &= \frac{1}{\bar{\sigma}}[(\sigma_x-\tfrac{1}{2}\sigma_\theta)\Delta\sigma_x + (\sigma_\theta-\tfrac{1}{2}\sigma_x)\Delta\sigma_\theta + 3\tau_{x\theta}\Delta\tau_{x\theta}]
\end{aligned} \tag{56}$$

$$K_0 = \frac{1-2v}{3E}, \quad K_1 = \frac{1}{E_t} - \frac{1}{E_s} \tag{57}$$

denote the equivalent stress and its increment. Then the relation eqn (55) can be written in the following matrix form

$$\begin{Bmatrix} \Delta\varepsilon_x \\ \Delta\varepsilon_\theta \\ \Delta\gamma_{x\theta} \end{Bmatrix} = \begin{bmatrix} C^d_{11} & C^d_{12} & C^d_{14} \\ C^d_{21} & C^d_{22} & C^d_{24} \\ C^d_{41} & C^d_{42} & C^d_{44} \end{bmatrix} \begin{Bmatrix} \Delta\sigma_x \\ \Delta\sigma_\theta \\ \Delta\tau_{x\theta} \end{Bmatrix} \tag{58}$$

in which

$$C^d_{11} = K_0 + \frac{1}{E_s} + K_1\left[\frac{2\sigma_x - \sigma_\theta}{2\bar{\sigma}}\right]^2, \quad C^d_{22} = K_0 + \frac{1}{E_s} + K_1\left(\frac{2\sigma_\theta - \sigma_x}{2\bar{\sigma}}\right)^2$$

$$C^d_{44} = 3\left[\frac{1}{E_s} + K_1\left(\frac{\tau_{x\theta}}{\bar{\sigma}}\right)^2\right], \quad C^d_{12} = C^d_{21} = K_0 - \frac{1}{2E_s} + K_1\frac{2\sigma_x - \sigma_\theta}{2\bar{\sigma}}\frac{2\sigma_\theta - \sigma_x}{2\bar{\sigma}}$$

$$C^d_{14} = C^d_{41} = 3K_1\frac{2\sigma_x - \sigma_\theta}{2\bar{\sigma}}\frac{\tau_{x\theta}}{\bar{\sigma}}, \quad C^d_{24} = C^d_{42} = 3K_1\frac{2\sigma_\theta - \sigma_x}{2\bar{\sigma}}\frac{\tau_{x\theta}}{\bar{\sigma}} \tag{59}$$

The incremental stress can also be expressed by incremental strain components

$$\begin{Bmatrix} \Delta\sigma_x \\ \Delta\sigma_\theta \\ \Delta\tau_{x\theta} \end{Bmatrix} = \begin{bmatrix} D^d_{11} & D^d_{12} & D^d_{14} \\ D^d_{21} & D^d_{22} & D^d_{24} \\ D^d_{41} & D^d_{42} & D^d_{44} \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_x \\ \Delta\varepsilon_\theta \\ \Delta\gamma_{x\theta} \end{Bmatrix} \tag{60}$$

where

$$[D^d] = [C^d]^{-1} \tag{61}$$

or by using the following relation

$$\bar{\varepsilon} = \frac{1}{\sqrt{3}}\sqrt{(\varepsilon_x - \varepsilon_\theta)^2 + \frac{3(\varepsilon_x + \varepsilon_\theta)^2}{k_0} + \gamma^2_{x\theta}}$$

$$\Delta\bar{\varepsilon} = \frac{1}{k}[k_1\Delta\varepsilon_x + k_2\Delta\varepsilon_\theta + k_3\Delta\gamma_{x\theta}] \tag{62}$$

the elements of $[D^d]$ can be expressed as

$$D^d_{11} = \tfrac{4}{3}E_s\left(C_5 - \frac{3k_1^2}{4k}C_6\right), \quad D^d_{22} = \tfrac{4}{3}\left(C_5 - \frac{3k_2^2}{4k}C_6\right)$$

$$D^d_{44} = \frac{E_s}{3}\left(1 - \frac{3k_3^2}{k}C_6\right), \quad D^d_{21} = D^d_{12} = \tfrac{2}{3}E_s\left(C_4 - \frac{3k_1k_2}{2k}C_6\right)$$

$$D^d_{14} = D^d_{41} = \frac{k_1k_3}{k}(E_t - E_s), \quad D^d_{24} = D^d_{42} = \frac{k_2k_3}{k}(E_t - E_s) \tag{63}$$

in which

$$C_1 = \frac{E}{1-2\nu}+\frac{4}{3}E_s,\quad C_2 = \frac{E}{1-2\nu}+\frac{2}{3}E_s,\quad C_4 = 1-\frac{2E_s}{C_1},\quad C_5 = 1-\frac{E_s}{C_1}$$

$$C_6 = \frac{E_s-E_t}{E_s},\quad k_0 = \left[1+\frac{4(1-2\nu)}{3E}E_s\right]^2,\quad k = 1-\frac{(E_s-E_t)(\sigma_x+\sigma_\theta)^2}{3C_1\bar{\sigma}^2}$$

$$k_1 = \frac{1}{C_1\bar{\sigma}}[C_2\sigma_x-\tfrac{2}{3}E_s\sigma_\theta],\quad k_2 = \frac{1}{C_1\bar{\sigma}}[C_2\sigma_\theta-\tfrac{2}{3}E_s\sigma_x],\quad k_3 = \frac{\tau_{x\theta}}{\bar{\sigma}} \tag{64}$$

Choosing the stress function the same as that in eqn (42) and following the same procedure as that in the previous section, one obtains the following expression for the critical buckling stress of a thin-walled cylinder:

$$\sigma_{cr} = |\sigma_x|_{min} = \frac{h}{\sqrt{3}R}\sqrt{\frac{D^d_{11}}{C^d_{22}}},\quad \text{or}\quad \frac{R}{h} = \frac{1}{\sqrt{3}\sigma_{cr}}\sqrt{\frac{D^d_{11}}{C^d_{22}}} \tag{65}$$

ANALYTICAL RESULTS AND EXPERIMENTAL VERIFICATION

The critical stresses σ_{cr} of the thin-walled cylinders subjected to longitudinal compressive loading were analyzed with the derived relations and compared with the experimental data (Shen and Han, 1981). These cylinders were made of aluminum alloys AMГ and П1T. By choosing $n = 3$, the corresponding material constants were determined as follows:

AMГ $C_{1,2,3} = (8.065*10^4, 8.654*10^3, 1.686*10^3)$ MPa
$\alpha_{1,2,3} = 1987, 227.8, 18.1$
$f(z) = 1$

П1T: $C_{1,2,3} = (1.003*10^6, 1.597*10^4, 2.160*10^2)$ MPa
$\alpha_{1,2,3} = 4603, 411.7, 17.6$
$f(z) = 1$

The elastic modulus E and Poisson's ratio ν of both materials are 73.0 GPa and 0.28; respectively. The tensile σ-ε relations of the two materials determined by the above material constants are shown in Fig. 2(a) and Fig. 2(b), respectively. It is seen that the fitting curves are in satisfactory agreement with the experimental data (Shen and Han, 1981).

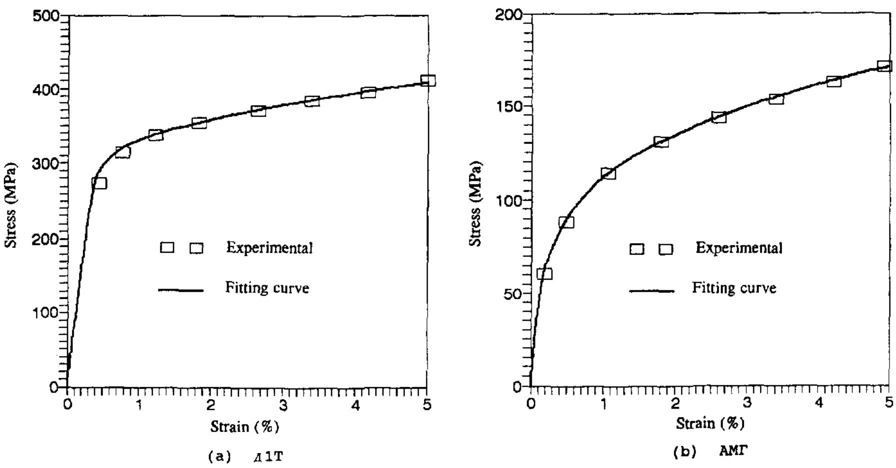

Fig. 2. Tensile stress and strain relations.

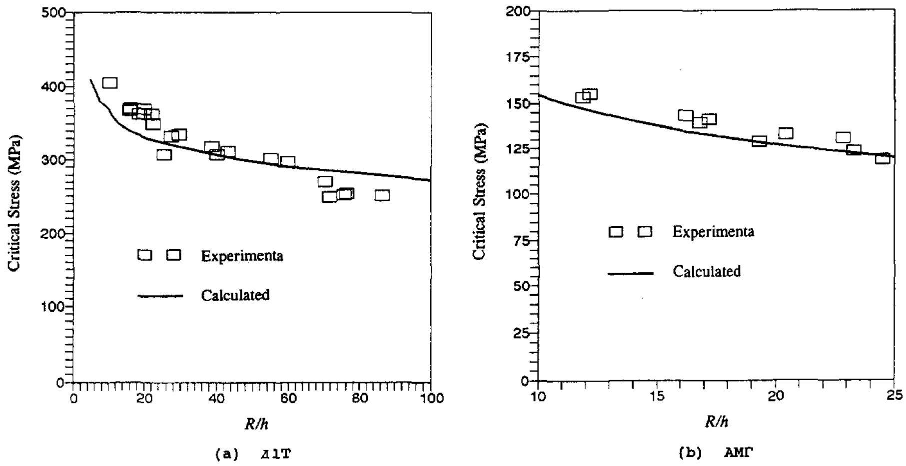

Fig. 3. The relations between σ_{cr} and R/h analyzed by the finite form of NCP and experimental data.

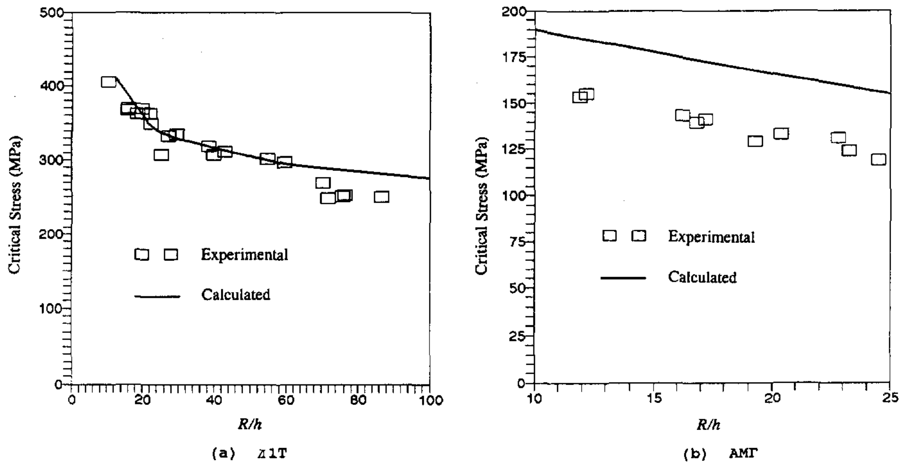

Fig. 4. The relations between σ_{cr} and R/h analyzed by the incremental form of NCP and experimental data.

Figures 3 and 4 show the variation of critical stress σ_{cr} against R/h (the characteristic size of the thin-walled cylinders) calculated respectively by the finite and incremental forms of NCP and the experimental data (Shen and Han, 1981), the ratio of the length of the cylinders L vs the radius R is around 3 for specimens, which is much larger than the wave length of the buckled cylinders. For the cylinders made of AMΓ the relation between σ_{cr} and R/h predicted by the finite form of NCP is very close to the experimental result (see Fig. 3(b)), while the incremental form of NCP overpredicts the critical stress σ_{cr} (the largest error is about 25%, see Fig. 4(b)). For the cylinders made of Π1T the finite form of NCP predicts a little more conservatively σ_{cr} when R/h is below 70 (see Fig. 3(a)), in this range, the critical stress σ_{cr} predicted by the incremental form of NCP also fits the experimental results well (see Fig. 4(a)). In the range where $R/h > 70$ it is seen that either finite or incremental form of NCP overpredicts the critical stress σ_{cr} (see Fig. 3(a) and Fig. 4(a)). It is seen that when R/h is larger than 70 the corresponding σ_{cr} almost suddenly drops from 300 MPa to around 250 MPa. Compared with the tensile stress and strain relation, it is easily found that in this region there is a very strong variation or reduction of the tangent modulus (see Fig. 2(a)). Noticing that in this region, the wall of the cylinder becomes very thin so that the structure becomes very sensitive to local material or structural defects, this abnormal reduction of σ_{cr} may, therefore, be mainly attributed to the effect of local structural or material defects, and a proper consideration of the effect may be helpful to produce a more reasonable prediction.

DISCUSSION AND CONCLUSION

Discussion

A haunted problem in plastic buckling analysis is that the incremental theory of plasticity, which is generally thought to be able to generate a more realistic result, fails to give a better prediction than the deformation theory of plasticity. This problem is serious and greatly slows down the progress in plastic buckling analysis. In the past several decades, a lot of work has concentrated on this paradox and many explanations and modifications have been suggested (Batdof, 1949; Sewell, 1963; Cristoffenson and Hutchinson, 1979), but few of them are general enough to work well in many cases. Onat and Drucker (1953) pointed out that by taking into account the effects of the initial microdefects, the buckling loads would have a great reduction. This explanation was supported by Hutchinson (1973), Hutchinson and Budiansky (1974), but some further work (Gellin, 1979; Roche, 1986; Bushnell, 1982) showed that there still exist some phenomena that can hardly be illustrated by this kind of explanation.

On the other hand, it was found in a strain-controlled biaxial material test that in the plastic range an abrupt change in the direction of strain path results in a reduction of the load-carrying capability of the material or the material instability (Ohashi and Tanaka, 1983; Ohashi *et al.*, 1975). This is an important concept, which, combined with the concept of initial nonhomogeneity caused by microdefects, may provide some available information to plastic buckling analysis. Microscopically speaking (Ohashi and Tanaka, 1983; Ohashi *et al.*, 1975), material instability is induced by a change in microscopic structure, in other words, the accumulated dislocation formed during the preceding deformation is remobilized by an abrupt change in strain path and this causes a release of the energy that is stored in the microstructure (as the energy stored in the spring C_r, see Fig. 1 and the illustration in part 2) and reduces the external work needed for dislocation movement. Macroscopically the material partly loses the load-carrying capability. NCP is able to describe this kind of material behavior. Figure 5 shows the response of the adopted two kinds of materials under strain-controlled loading with abrupt changes in strain paths. It is seen that material instability occurs at each change in strain trajectory. This phenomenon is in qualitative agreement with the experimental results reported by Ohashi and Tanaka (1983) and Ohashi *et al.* (1975). Although what is described in Fig. 5 is a pure material property, the information provided may be available to illustrate the "paradox". Compared with П1T, the reduction of the load-carrying capability of AMΓ caused by material instability is more severe so that the corresponding critical stress σ_{cr} predicted by the incremental form of the constitutive relation is higher without a proper consideration of this reduction. Since nonhomogeneity caused by microdefects inevitably exists, local instability might occur in a structure earlier than that with the perfect material, as used in the analysis.

In some other work (Sewell, 1963; Cristoffenson and Hutchinson, 1979), it was argued that the corner at loading surface plays an important role in plastic buckling because the

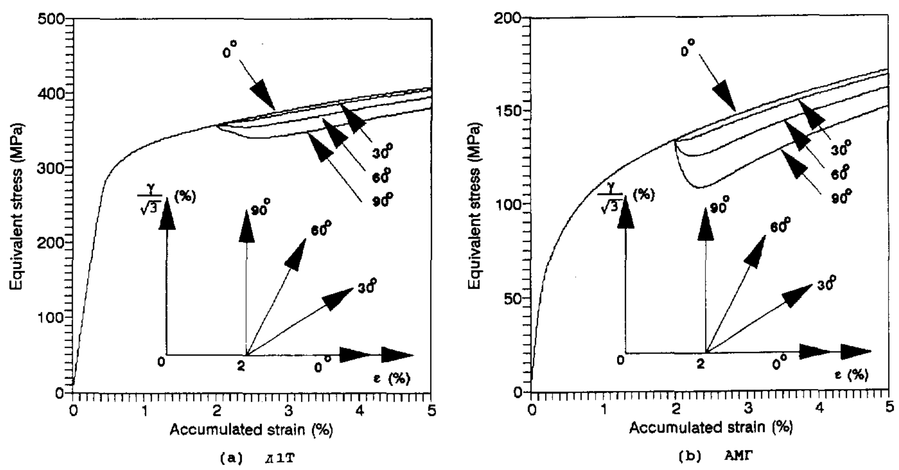

Fig. 5. Prediction of material instability.

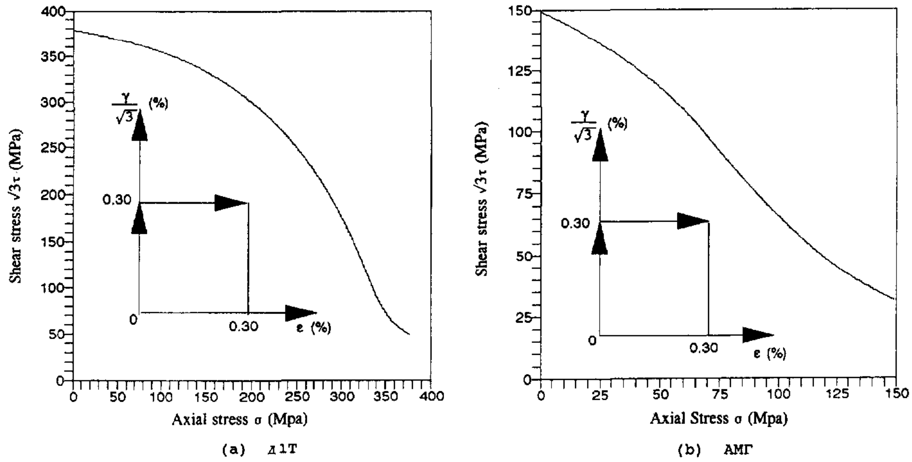

Fig. 6. Stress trajectory corresponding to pure shear strain following pure tensile strain.

buckling stress is very sensitive to the out normal of the local loading surface. The adopted constitutive relation can well describe the effect of the corner in loading path. Figure 6 shows the stress trajectories of the adopted materials subjected to the path of axial straining following shear straining ranged around 3%. It is seen that for the given strain path, the shear stress decreases once the tensile strain occurs. One can, therefore, imagine that at the critical state of the buckling of a thin-walled cylinder, given a disturbing strain perpendicular to the original strain direction, the original stress component can hardly hold constant without a marked increase of the corresponding strain component. The consideration of this material property may also be of help for plastic buckling analysis.

Although the NCP is able to describe phenomena such as material instability and the corner in the loading path, which may be of benefit to settle the paradox in plastic buckling analysis, it is seen that the incremental form of the adopted constitutive relation still overpredicts the critical buckling load in some cases. It may partly be attributed to that when disturbance occurs, on one hand, the material instability occurs, which partly loses the load-carrying-capability; on the other hand, both the experimental result (Ohashi and Tanaka, 1983; Ohashi *et al.*, 1975) and the theoretical analysis show that at the instant when material instability occurs, the material response is almost purely elastic, which leads to an increase of the stiffness of the system, which, in turn, leads to a higher critical stress. The finite form of NCP cannot describe the phenomenon of material instability and the corresponding increase of the stiffness of material and its system, which, on the contrary, provides a more reasonable critical stress.

Conclusion

In this work the incremental and finite forms of the NCP are applied to the analysis of the plastic buckling of thin-walled cylinders subjected to longitudinal compressive load. Compared with the experimental data, it is seen that both forms of the adopted constitutive relation can produce satisfactory prediction for the critical stress of the cylinders made of aluminum ΠlT in a wide range of the characteristic size R/h of the cylinder. But for the cylinders made of aluminum AMΓ the incremental form of NCP produces less satisfactory prediction than the finite form of NCP. Generally, the critical stress predicted by the finite form of the NCP is smaller than that by the incremental form and closer to the experimental data.

The capability of NCP in the description of material instability and the corner of loading path is discussed. Although the incremental form of NCP is able to describe material instability and the response of material under an abrupt strain disturbance, it still overpredicts the critical plastic buckling stress in some cases. The discrepancy between the reduction of the load-carrying capability caused by material instability and the increase of the stiffness of the system, as well as the nonhomogeneity due to the existence of microdefects may partly account for the existing paradox.

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